COMITATO NAZIONALE PER L'ENERGIA NUCLEARE
Laboratori Nazionali di Frascati

LNF - 62/105
10 Dicembre 1962

C. Pellegrini: EQUATIONS OF MOTION OF SPINNING PARTICLES IN THE TETRAD THEORY OF GRAVITATION.

Nota interna: n. 172

Servizio Documentazione
dei Laboratori Nazionali di Frascati del CNEN
Casella Postale 70 - Frascati (Roma)

Laboratori Nazionali di Frascati del CNEN
Servizio Documentazione

LNF-62/105

Nota interna: n° 172
10 Dicembre 1962

C. Pellegrini: EQUATIONS OF MOTION OF SPINNING PARTICLES IN THE TETRAD THEORY OF GRAVITATION.

ABSTRACT.

Equations of motion of spinning particles are derived from the generalized Bianchi identities obtained by Pellegrini and Plebanski in the framework of the tetrad theory of gravitational fields.

1) The use of tetrads as field variables in the theory of gravitation has been considered by several authors because of the interesting possibilities it offers.

As first pointed out by Weyl[1] the theories based on this hypothesis differ from the usual Einstein theory when matter is present, although the gravitational Lagrangian is still assumed to be the scalar curvature R, which is invariant under the group of arbitrary Lorentz-rotations of the tetrads.

The requirement that even the matter Lagrangian be invariant under the same group has been used by Sciama[2] to give a dynamical definition of the spin.

Utiyama[3] and Kibble[4] were able to show that the existence of the gravitational field, described by means of tetrads, can be related to the invariance of the matter Lagrangian under the group of x-dependent Lorentz coordinate transformations.

The possibility of obtaining, by using tetrads, a gravitational energy-momentum complex $\Theta_{\alpha\beta}$ such that the energy-density Θ_{00} is a scalar under purely spatial coordinate transformations, has been discovered by Møller[5] and used by him in connection with the problem of the definition of the gravitational waves energy[6].

In a recent work Pellegrini and Plebanski[7] have derived the tetrad theory of gravitational fields from an Action Principle, using a Lagrangian function of the tetrads and their first derivatives only and in which a new term is summed to the scalar curvature. This theory gives a correct answer to the three experimental tests of General Relativity and furthermore offers an easy way to introduce spinor fields in a curved space-time. This definition of spinors is such that the covariant derivative of a spinor is simply the partial derivative.

As a consequence of this fact, the matter tensor of a Dirac field contains a skew-symmetric part which enters explicitely into the field equations and into the Bianchi identities. These assume now the new form(x)

(1-1) $$T^{\alpha\beta}{}_{;\beta} - \gamma_{[\beta\gamma]}{}^{\alpha}\, T^{[\beta\gamma]} = 0$$

As it is well known the equations of motion of a point-like mass can be derived from the standard Bianchi identities $T^{\alpha\beta}{}_{;\beta} = 0$.

In this paper we want to derive the same equations starting from (1-1). Clearly the equations that we will obtain must differ from the geodesic equations due to the presence of the new term $\gamma_{[\beta\gamma]}{}^{\alpha}\, T^{[\beta\gamma]}$ in (1-1). Since $T^{[\alpha\beta]}$ is closely connected with the spin angular momentum, we expect to get from (1-1) the equations describing the motion of a spinning particle.

These will be derived in section 3, while section 2 is devoted mainly to the definition of $T^{\alpha\beta}$ and section 4 to a short discussion of the results obtained and their comparison with those of other authors.

2) The notations used in this work are those of reference[7].

(x) - The symmetric and antisymmetric parts of a tensor $T^{\alpha\beta}$ are indicated by $T^{(\alpha\beta)}$ and $T^{[\alpha\beta]}$ respectively.

In particular, the signature -2 is assumed for the metric.

$\hat{\ }$ The tensor $\gamma_{[\alpha\beta]\gamma}$ is given, in terms of the tetrad field $g^{\hat{\alpha}}_{\alpha}$ (x), by

$$\gamma_{[\alpha\beta]\gamma} = g^{\hat{\alpha}}_{\alpha}\, g_{\hat{\alpha}\beta;\gamma} \quad . \tag{2-1}$$

We consider now a test particle whose world-line is Γ. Let

$$\xi^{\alpha} = \xi^{\alpha}(\lambda)$$

be the equations of Γ and

$$t^{\alpha\beta}(\lambda)$$

the energy-momentum tensor of the particle, λ being an arbitrary parameter defined along Γ.

An energy-momentum tensor density field can be defined in terms of the line a tensor $t^{\alpha\beta}(\lambda)$ by

$$T^{\alpha\beta}(x) = \int_{-\infty}^{+\infty} d\lambda\, t^{\alpha\beta}(\lambda)\, \delta_{(4)}(x - \xi(\lambda)) \, . \tag{2-2}$$

Integrating (2-2) over a three-dimensional surface σ orthogonal to Γ and intersecting Γ in a point of coordinates ξ_{σ} we obtain

$$t^{\alpha\beta}(\xi_{\sigma}) = \int_{\sigma} T^{\alpha\beta}\, d\sigma \, , \tag{2-3}$$

where $d\sigma = u^{\alpha} d\sigma_{\alpha}$ is the invariant volume element of σ and

$$u^{\alpha} = \frac{d\xi^{\alpha}}{ds}$$

is the four-velocity of the particle, ds being the arc length. Equation (2-3) shows that the quantity on the right-hand side is a line tensor, a result which will be useful later on.

3) To derive the equations of motion the tensor density defined by (2-2) is substituted in (1-1), which can be written as

$$T^{\alpha\beta}{}_{,\beta} + \Gamma^{\alpha}_{\beta\gamma}\, T^{(\beta\gamma)} - \gamma_{[\beta\gamma]}{}^{\alpha}\, T^{[\beta\gamma]} = 0 \quad . \tag{3-1}$$

From this one obtains

(3-2) $$(x^{\varrho} T^{\alpha\beta})_{/\beta} = T^{\alpha\varrho} - x^{\varrho}\left\{\Gamma_{\beta\gamma}^{\alpha} T^{(\beta\gamma)} - \gamma_{[\beta\gamma]}^{\alpha} T^{[\beta\gamma]}\right\} = 0$$

Integrating (3-1), (3-2) over the three-dimensional surface σ orthogonal to u^{α} and specializing to the frame of reference where the particle is at rest, one obtains, by means of Gauss theorem,

(3-3) $$\frac{d}{dx^0}\int_{x_0=\mathrm{cost}} T^{\alpha 0} d^3x + \Gamma_{\beta\gamma}^{\alpha}(\xi_{x_0}^{\varrho}) \int_{x_0=\mathrm{cost}} T^{(\beta\gamma)} d^3x - \gamma_{[\beta\gamma]}^{\alpha}(\xi_{x_0}^{\varrho}) \int_{x_0=\mathrm{cost}} T^{[\beta\gamma]} d^3x = 0$$

(3-4) $$\frac{d\xi^{\beta}}{dx^0}\int_{x_0=\mathrm{cost}} T^{\alpha 0} d^3x = \int_{x_0=\mathrm{cost}} T^{\alpha\beta} d^3x ,$$

having made use of the fact that only the intersection of σ and Γ contributes to the integral of $T^{\alpha\beta}$. In an arbitrary frame of reference (3-3), (3-4) become

(3-5) $$\frac{d}{ds}\int_{\sigma} T^{\alpha\beta} d\sigma_{\beta} + \Gamma_{\beta\gamma}^{\alpha} \int T^{(\beta\gamma)} d\sigma - \gamma_{[\beta\gamma]}^{\alpha}(\xi_{\sigma}^{\varrho}) \int_{\sigma} T^{[\beta\gamma]} d\sigma = 0$$

(3-6) $$u^{\beta}\int_{\sigma} T^{\alpha\gamma} d\sigma_{\gamma} = \int_{\sigma} T^{\alpha\beta} d\sigma .$$

These can be written in a simpler way, since, multiplying (3-6) by u_{β} and using (2-3) one obtains

(3-7) $$\int_{\sigma} T^{\alpha\gamma} d\sigma_{\gamma} = u_{\gamma}\int_{\sigma} T^{\alpha\gamma} d\sigma \equiv u_{\gamma}\, t^{\alpha\gamma}(\xi_{\sigma})$$

a result which allows us to put (3-5), (3-6) in the form

(3-8) $$\frac{d}{ds}\left(u_{\beta}\, t^{\alpha\beta}(\xi)\right) + \Gamma_{\beta\gamma}^{\alpha}(\xi)\, t^{(\alpha\beta)}(\xi) - \gamma_{[\beta\gamma]}^{\alpha}(\xi) = 0 \quad .$$

(3-9) $$u^{\beta} u_{\gamma}\, t^{\alpha\gamma}(\xi) = t^{\alpha\beta}(\xi) ,$$

Equation (3-9) determines completely the structure of $t^{\alpha\beta}$. In fact multiplying by u_α and introducing the mass

(3-10) $$m = u_\alpha u_\beta t^{\alpha\beta} ,$$

it follows that

(3-11) $$u_\alpha t^{\alpha\beta} = m u^\beta .$$

When $t^{\alpha\beta}$ is symmetric (3-9), (3-11) give

$$t^{\alpha\beta} = m u^\alpha u^\beta ,$$

In the general case one has

$$u_\alpha t^{(\alpha\beta)} + u_\alpha t^{[\alpha\beta]} = m u^\beta$$

and substituting in (3-9)

(3-12) $$t^{\alpha\beta}(\xi) = m u^\alpha u^\beta + 2 u^\beta u_\gamma t^{[\alpha\gamma]} ,$$

In analogy with the case of special relativity we can now relate $t^{[\alpha\beta]}$ to the derivative of the antisymmetric spin tensor S :

(3-13) $$\frac{\delta}{\delta s} S^{[\alpha\beta]} = 2 t^{[\alpha\beta]} = 2 \int_\sigma T^{[\alpha\beta]} d\sigma ,$$

where the derivative is performed along Γ.
Eventually $t^{\alpha\beta}(\xi)$ can be written as

(3-14) $$t^{\alpha\beta}(\xi) = m u^\alpha u^\beta + u^\beta u_\gamma \frac{\delta S^{[\alpha\beta]}}{\delta s} .$$

The fact that the Bianchi identities impose some conditions on the energy tensor has been known for a long time(8). Eq. (3-14) shows how, in our theory, the intrinsic spin modifies the usual form, $t^{\alpha\beta}(\xi) = m u^\alpha u^\beta$.

Substituting (3-14) in (3-8) we obtain the equations of motion of the particle

(3-15) $$\frac{\delta}{\delta s}\left\{ m u^\alpha + u_\gamma \frac{\delta s^{[\alpha\gamma]}}{\delta s}\right\} - \gamma^{\alpha}_{[\beta\gamma]}\, u^\gamma u_\rho \frac{\delta s^{[\beta\rho]}}{\delta s} = 0 \quad ,$$

while the antisymmetric part of (3-14) gives us the equations of motion of the spin

(3-16) $$\frac{\delta s^{[\alpha\beta]}}{\delta s} = u_\gamma \left\{ u^\beta \frac{\delta s^{[\alpha\gamma]}}{\delta s} - u^\alpha \frac{\delta s^{[\beta\gamma]}}{\delta s} \right\} .$$

Defining the momentum of the particle as

(3-17) $$P^\alpha = m u^\alpha + u_\gamma \frac{\delta s^{[\alpha\gamma]}}{\delta s}$$

eq. (3-15), (3-16) can be rewritten as

(3-18) $$\frac{\delta P^\alpha}{\delta s} = \gamma^{\alpha}_{[\beta\gamma]}\, u^\gamma P^\beta$$

(3-19) $$\frac{\delta s^{[\alpha\beta]}}{\delta s} = u^\beta P^\alpha - u^\alpha P^\beta \quad ,$$

As usual to these equations we must add a set of supplementary conditions, since four of them are just identities.

In the ordinary theory of a spinning particle this corresponds to the freedom in the motion of the pole with respect to which the angular momentum is evaluated.

If a particle is endowed with intrinsic spin, the missing conditions must follow from its inner dynamics. Various possibilities have been suggested because of their simplicity[9].

For instance one can choose the condition

$$u_\beta\, s^{[\alpha\beta]} = 0$$

which assures us that

$$s_{\alpha\beta}\, s^{\alpha\beta} = \text{const},$$

and that in the case of special relativity P^α does not become simply parallel to u^α.

4) The motion of spinning particles has been already y a studied by Papapetrou[10], in the framework of the Einstein theory, and by Sciama[2], who relates the spin to the antisym

metric part of $T^{\alpha\beta}$ and makes use of tetrads.

In the work of Papapetrou the spin is defined as the dipole moment of the matter distribution and the contribution of all higher multipoles is neglected. Since we do not need to do this approximation and can simply define the spin by means of $T^{[\alpha\beta]}$, it is noteworthy that the equations for the spin motion obtained by Papapetrou and by us are identical.

It is not so for the other equations, which differ from those of Papapetrou and Sciama for the "force" term.

The spin equations (3-19), which in the limit of flat space-time simply give the conservation of the total angular momentum, i.e.

$$S^{\alpha\beta} + \xi^{\alpha} p^{\beta} - \xi^{\beta} p^{\alpha} = \text{Const},$$

have been extensively investigated in the general case by Schiff[11], who showed that they can be used to provide an experimental test of General Relativity.

This is not the case for eq. (3-18), or for the analogous equations of Papapetron and Sciama, since the deviations from a geodesic due to the "force" terms are far too small to be observable with our present experimental techniques. This fact does not allow the use of (3-18) to test the conservation equation (1-1).

At last we want to remind that, as pointed out by Sciama[2], the fact that, according to (3-18) spinning particles do not follow a geodesic line invalidates in general the principle of equivalence in its weak form[12], since it violates the requirement that any gravitational effect be locally irrelevant, even for electrons and protons.

The author whishes to thank Professor B. Bertotti and Professor F.A.E. Pirani for many useful discussions.

References.

(1) - H. Weyl, Phys. Rev. 77, 699 (1950).

(2) - D.W. Sciama, in Recent developments in general relativity, pag. 415, Warsaw 1962.

(3) - R. Utiyama, Phys. Rev. 101, 1597 (1956).

(4) - T.W.B. Kibble, J. Math. Phys. 2, 212 (1961).

(5) - C. Møller, Mat. Fys. Skr. Dan. Vid. Selsk. 1, no.10 (1961).

(6) - C. Møller, Phys. Lett. 3, 329 (1963).

(7) - C. Pellegrini, J. Plebanski, Mat. Fys. Skr. Dan. Vid. Selsk. 2, no. 4 (1963).

(8) - P. Havas and J.N. Goldberg, Phys. Rev. 128, 398 (1962)

(9) - See for instance A. Papapetrou and E. Corinaldesi, Proc. Roy. Soc. 209, 259 (1951), and F.A.E. Pirani, Acta Phys. Polon. 15, 389 (1956).

(10) - A. Papapetrou, Proc. Roy. Soc. 209, 248 (1951).

(11) - L. Schiff, Phys. Rev. Letters 4, 215 (1960).

(12) - See for instance the paper by B. Bertotti, D. Brill, R. Krotkov, in Gravitation: an introduction to current research, edited by L. Witten, New York 1962.